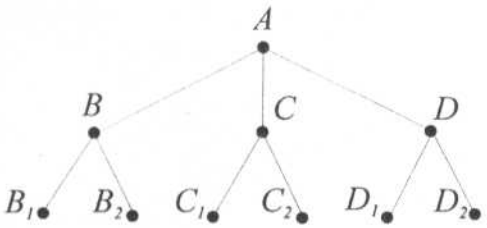
SAMPLES FROM "Fairyland's Problems Book"

19. The Ministry of Transportation of Camelot asked an airline to create flight connections between the major cities of the country such that any city would have at most 3 departing flights and you could reach any other city with no more than one change of flight. What is the maximum number of cities for which these connections could be designed?

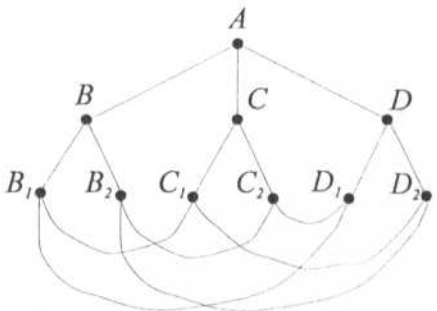
Solution:

From any arbitrary city A we can reach at most 3 other cities (B, C, D). One of the flights originating from B is a return flight to A, so at most 2 new cities could be reached from B. (B_1, B_2). Similarly, from each city C and D there are at most 2 new destinations.



There are no more possible cities, because you wouldn't be able to get to a new city from A with only one change. So we can have at most 10 cities.

Is it possible to make the required connections between all 10 cities? First we need to make sure that the 10 above-mentioned cities are distinct. On the other hand we have to be able to create appropriate connections between the cities B_1, B_2, C_1, C_2, D_1, D_2. If we take into consideration that from each of these cities we can originate at most 2 new flights and that from B_1, B_2 we still have to get to C and D (with at most one change) then there aren't many choices to make. One possible solution can be seen in the illustration. Therefore we can make the required connections between 10 cities and no more.



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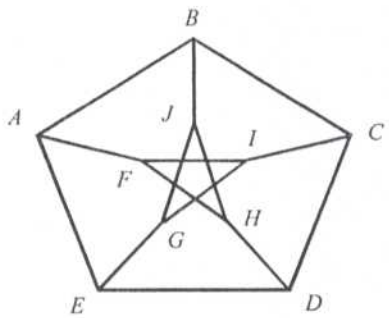
Note I:

There is a more symmetrical representation of the solution:



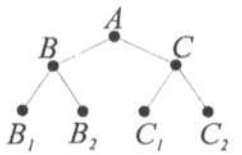
Márton Hablicsek

Teacher's note:

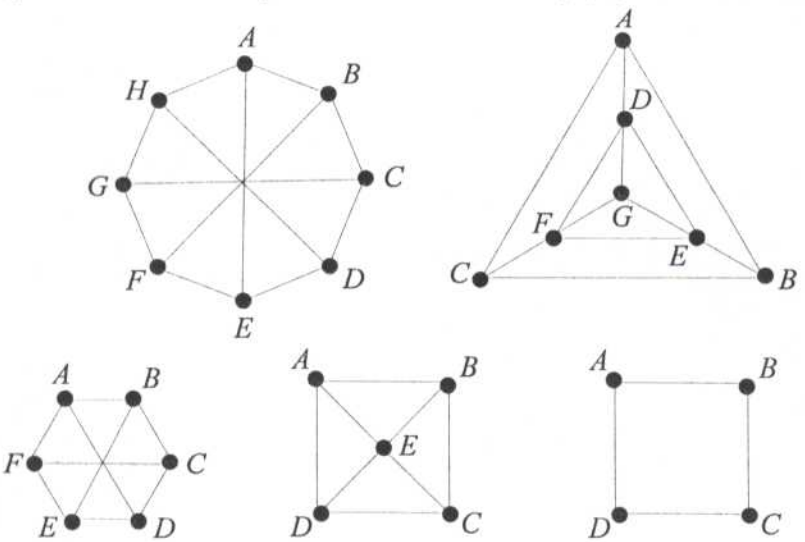
The graph constructed in solving the problem called the Petersen graph is unusually symmetrical. In the last illustration, the graph has a rotational symmetry of 108° around its center. Furthermore, any reflection and rotation that would map the ABCDE regular pentagon onto itself would also map the graph onto itself. It would be worth trying to draw this graph other ways, with some other types of symmetry. Is it possible, for example to draw it with rotational symmetry of 120° ? With point symmetry? With rotational symmetry of 90° ?

Note II:

Let's continue the original example! Could it be 9 cities? If from each city there are 3 originating flights, then there would be 3·9/2 = 13.5 flights, because we counted every flight twice (there and back). If from a city (A) there were no more than 2 flights
(to B and C), then at most 7 cities would be possible. Therefore 9 cities are not possible.



It is possible to make the required connections between 8, 7, 6, 5 and 4 cities. For example like this:



For fewer cities the problem doesn't have a lot of meaning.

Kinga Makovi and Dorottya Nagy

36. Old King Cole II hated his predecessor, Old King Cole I, so much that he prohibited the usage of the numeral 1. In his country, one had to count like this: 2, 3, 4, 5, 6, 7, 8, 9, 20, 22,...What number did they use in Old King Cole II's country to replace 1998? In other words, which is the 1998th number in our sequence?

Solution I:
Let's study the intervals to see which numbers were kept and thrown out by Old King Cole II.

Interval	Good ones	The total of the good ones	Bad ones	Total of the bad ones	
1-9	8	8	1	1	
10-19	0	8	10	11	
20-29	9	17	1	12	
30-99	63	80	7	19	
100-199	0	80	100	119	
200-209	9	89	1	120	
210-299	72	161	18	138	
300-999	567	728	133	271	
1000-1999	0	728	1000	1271	
2000-2999	729	1457	271	1542	
3000-3099	81	1538	19	1561	
3100-3199	0	1538	100	1661	
3200-3299	81	1619	19	1680	
3300-3399	81	1700	19	1699	
3400-3499	81	1781	19	1718	
3500-3599	81	1862	19	1737	
3600-3699	81	1943	19	1756	
3700-3719	9	1952	11	1767	
3720-3729	9	1961	1	1768	
3730-3739	9	1970	1	1769	
3740-3749	9	1979	1	1770	
3750-3759	9	1988	1	1771	
3760-3769	9	1997	1	1772	
3770	1	1998	0	1772	

Since the total of the good ones has to be 1998, the number at the bottom of the first column, 3770 is the solution.

Dávid Kutasi

Solution II:

Old King Cole III mocked his predecessor Old King Cole II's law. He decided to annul the law. But his countrymen were already used to the prohibition, so Old King Cole III asked his wizard to help him. The wizard suggested subtracting 1 from every digit other than 0.

Old King Cole II's digits	Old King Cole III's digits
0	0
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8

Notice that Old King Cole III's number system is exactly the base 9 system. In it the 1998th number is nothing else than 1998, just in the base 9 system. So let's rewrite 1998 in base 9:

9^3	9^2	9	1
729	81	9	1
2	6	6	0

Therefore in Old King Cole III's country, they used 2660 to replace 1998. Now let's convert the digits of 2660 to Old King Cole II's digits!

2	6	6	0
3	7	7	0

The number we are looking for is 3770.

Dávid Kutasi and Máté Juhász